

Lecture 21: Master Equation: Examples

Last lecture we introduced the Lindblad form of the master equation, the most general Markov equation consistent with CP-maps:

$$\frac{d\hat{\rho}}{dt} = \frac{-i}{\hbar} [\hat{H}, \hat{\rho}] + \mathcal{L}_{\text{relax}}[\hat{\rho}]$$

$$\mathcal{L}_{\text{relax}} = \sum_{\mu} \left[-\frac{1}{2} (\hat{L}_{\mu}^{\dagger} \hat{L}_{\mu} \hat{\rho} + \hat{\rho} \hat{L}_{\mu}^{\dagger} \hat{L}_{\mu}) + \hat{L}_{\mu} \hat{\rho} \hat{L}_{\mu}^{\dagger} \right]$$

The set $\{\hat{L}_{\mu}\}$ are the Lindblad "jump operators"

with $\gamma_{j \rightarrow j'}^{\mu} = |\langle j' | \hat{L}_{\mu} | j \rangle|^2$ the transition rate from $|j\rangle \rightarrow |j'\rangle$ according to a process μ .

Example 1: Two-level atom in a reservoir of black-body radiation

A canonical problem in quantum optics is a two level atom coupled to thermal reservoir of black-body radiation. Including the quantum fluctuations, this also include spontaneous emission in the vacuum (zero-temperature reservoir)

The Lindblad equation follow from the usual system+environment Born-Markov approximation, with

$$\hat{H}_{\text{total}} = \underbrace{\frac{\hbar\omega_A}{2} \hat{\sigma}_z}_{H_S} + \underbrace{\sum_k \hbar\omega_k a_k^\dagger a_k}_{H_E} + \hbar \underbrace{\sum_k (g_k a_k \sigma_+ + g_k^* a_k^\dagger \sigma_-)}_{H_{SE}}$$

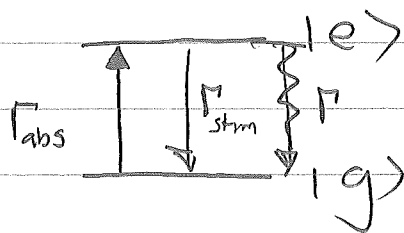
The field is the 'environment' in a thermal state

$$\hat{\rho}_E(0) = \prod_k \frac{e^{-\beta \hbar \omega_k a_k^\dagger a_k}}{Z_k} = \prod_k \frac{\bar{n}_k^{n_k}}{(\bar{n}_k + 1)^{n_k + 1}}, \quad |n_k\rangle \langle n_k|$$

where $\beta = \frac{1}{k_B T}$, $Z_k = \frac{1}{1 - e^{-\beta \hbar \omega_k}}$, $\bar{n}_k = \frac{1}{e^{\beta \hbar \omega_k} - 1}$

Note: at $\beta \rightarrow \infty$ ($T=0$) $\hat{\rho}_E(0) \rightarrow |vac\rangle \langle vac|$

The interaction of the atom and field leads to absorption + emission



Γ_{abs} = absorption rate

Γ_{stim} = stimulated emission rate

Γ = spontaneous emission rate

According to the Einstein A-B relations

$$\Gamma_{\text{abs}} = \Gamma_{\text{stim}} = \bar{n} \Gamma$$

where $\bar{n} = \bar{n}(\omega_{eg}) = \frac{1}{e^{\beta \hbar \omega_{eg}} - 1}$

We thus have two Lindblad operators defined by

$$\Gamma_{\text{abs}} = |\langle e | \hat{L}_{\text{abs}} | g \rangle|^2 = \bar{n} \Gamma \Rightarrow \hat{L}_{\text{abs}} = \sqrt{\bar{n} \Gamma} \hat{\sigma}_+$$

$$\Gamma_{\text{emiss}} = |\langle g | \hat{L}_{\text{emiss}} | e \rangle|^2 = (\bar{n} + 1) \Gamma \Rightarrow \hat{L}_{\text{emiss}} = \sqrt{(\bar{n} + 1) \Gamma} \hat{\sigma}_-$$

We thus have the Master Eqn for the Atom

$$\frac{d\hat{\rho}_A}{dt} = -\frac{i}{\hbar} [\hat{H}_A, \hat{\rho}_A] + \mathcal{L}_{\text{relax}}[\hat{\rho}_A]$$

$$\mathcal{L}_{\text{relax}}[\hat{\rho}_A] = -\frac{\Gamma}{2}(\bar{n}+1) \left(\hat{\sigma}_+ \hat{\sigma}_- \hat{\rho}_A + \hat{\rho}_A \hat{\sigma}_+ \hat{\sigma}_- - 2\hat{\sigma}_- \hat{\rho}_A \hat{\sigma}_+ \right) \\ -\frac{\Gamma}{2}\bar{n} \left(\hat{\sigma}_- \hat{\sigma}_+ \hat{\rho}_A + \hat{\rho}_A \hat{\sigma}_- \hat{\sigma}_+ - 2\hat{\sigma}_+ \hat{\rho}_A \hat{\sigma}_- \right)$$

$$\hat{H}_A = \hbar\omega_{eg} |e\rangle\langle e| = \hbar\omega_{eg} \hat{\sigma}_+ \hat{\sigma}_-$$

$$\Rightarrow \frac{d\hat{\rho}}{dt} = -i\omega_{eg} [|e\rangle\langle e|, \hat{\rho}_A] - \frac{\Gamma}{2}(\bar{n}+1) \left(\{ |e\rangle\langle e|, \hat{\rho} \} - 2|g\rangle\langle g| \rho_{ee} \right) \\ - \frac{\Gamma}{2}\bar{n} \left(\{ |g\rangle\langle g|, \hat{\rho} \} - 2|e\rangle\langle e| \rho_{gg} \right)$$

anti-commutator

Evolution of matrix elements

$$\frac{d}{dt} \rho_{ee} = \frac{d}{dt} \langle e|\hat{\rho}|e\rangle = \underbrace{-\Gamma(\bar{n}+1)}_{\text{emission}} \rho_{ee} + \underbrace{\Gamma\bar{n}}_{\text{absorption}} \rho_{gg}$$

$$\frac{d}{dt} \rho_{gg} = \frac{d}{dt} \langle g|\hat{\rho}|g\rangle = -\Gamma\bar{n} \rho_{gg} + \Gamma(\bar{n}+1) \rho_{ee}$$

Trace preserving $\frac{d}{dt} (\rho_{gg} + \rho_{ee}) = 0$

Steady State \Rightarrow detailed balance

$$\Rightarrow \frac{d}{dt} \hat{\rho} = 0 \quad \Rightarrow \quad \frac{\rho_{ee}}{\rho_{gg}} = \frac{\bar{n}}{\bar{n}+1} = \frac{(e^{\beta\hbar\omega_{eg}} - 1)^{-1}}{(e^{\beta\hbar\omega_{eg}} + 1) + 1} \\ = e^{-\beta\hbar\omega_{eg}} \quad \text{Boltzmann!} \quad \checkmark$$

Thus, in steady state the atom come to equilibrium with the bath, as expected

In fact, Einstein derived the spontaneous emission rate to get thermal equilibrium (see: Einstein A/B coefficients)

Decay of coherences (in the absence of coherent driving)

$$\begin{aligned}\frac{d}{dt} \rho_{eg} &= \frac{d}{dt} \langle e | \hat{\rho} | g \rangle \\ &= -i\omega_{eg} \rho_{eg} - \frac{\Gamma}{2} (2\bar{n} + 1) \rho_{eg}\end{aligned}$$

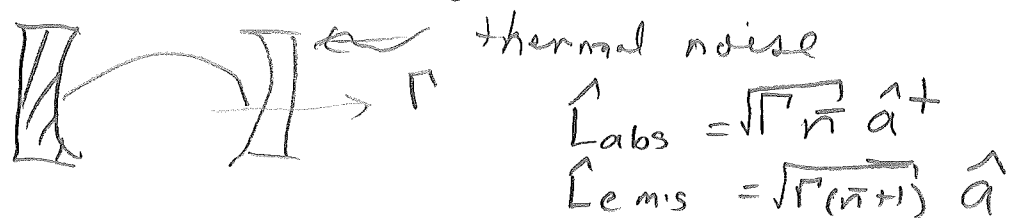
Decay of coherences $\gamma = \frac{\Gamma_e + \Gamma_g}{2}$
(in the absence of collisions)

Note: Without coherent drives
Separate equations of coherences and populations

Another example: Damped SHO

Given oscillator @ freq ω_0 coupled to a bath of thermal oscillators

E.g. mode in leaky cavity



$$\frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar} [\hat{H}, \hat{\rho}] - \frac{\Gamma}{2} (\bar{n} + 1) \left[\{ \hat{a}^\dagger \hat{a}, \hat{\rho} \} - 2 \hat{a} \hat{\rho} \hat{a}^\dagger \right] - \frac{\Gamma}{2} \bar{n} \left[\{ \hat{a} \hat{a}^\dagger, \hat{\rho} \} - 2 \hat{a}^\dagger \hat{\rho} \hat{a} \right]$$

Derived in same Born-Markov approx with

$$\hat{H} = \hbar \omega_0 \hat{a}^\dagger \hat{a} + \sum_k \hbar \omega_k \hat{a}_k^\dagger \hat{a}_k + \sum_k \left(g_k \hat{b}_k^\dagger \hat{a} + g_k^* \hat{b}_k \hat{a}^\dagger \right)$$

↑
linear coupling

In condensed matter literature known as "Caldeira-Leggett" model)

Consider the evolution of expectation values of observables:

$$\text{Aside: } \frac{d}{dt} \langle \hat{A} \rangle = \frac{d}{dt} \text{Tr}(\rho \hat{A}) = \text{Tr}\left(\frac{d\rho}{dt} \hat{A}\right)$$

$$\begin{aligned} \Rightarrow \frac{d}{dt} \langle \hat{A} \rangle &= -\frac{1}{2} \sum_{\mu} \text{Tr} \left(\hat{L}_{\mu}^{\dagger} \hat{L}_{\mu} \rho \hat{A} + \rho \hat{L}_{\mu}^{\dagger} \hat{L}_{\mu} \hat{A} - 2 \hat{L}_{\mu} \rho \hat{L}_{\mu}^{\dagger} \hat{A} \right) \\ &= -\frac{1}{2} \sum_{\mu} \text{Tr} \left[(\hat{A} \hat{L}_{\mu}^{\dagger} \hat{L}_{\mu} + \hat{L}_{\mu}^{\dagger} \hat{L}_{\mu} \hat{A} - 2 \hat{L}_{\mu}^{\dagger} \hat{A} \hat{L}_{\mu}) \rho \right] \end{aligned}$$

$$\boxed{\frac{d}{dt} \langle \hat{A} \rangle = -\frac{1}{2} \sum_{\mu} \left(\langle \hat{L}_{\mu}^{\dagger} [\hat{L}_{\mu}, \hat{A}] \rangle + \langle [\hat{A}, \hat{L}_{\mu}^{\dagger}] \hat{L}_{\mu} \rangle \right)}$$

For example, in damped SHO: mean excitation

$$\begin{aligned} \frac{d}{dt} \langle \hat{n} \rangle &= -\frac{\Gamma}{2} (\bar{n} + 1) \left(\langle \hat{a}^{\dagger} [\hat{a}, \hat{n}] \rangle + \langle [\hat{n}, \hat{a}^{\dagger}] \hat{a} \rangle \right) \\ &\quad + \frac{\Gamma}{2} \bar{n} \left(\langle \hat{a} [\hat{a}^{\dagger}, \hat{n}] \rangle + \langle [\hat{n}, \hat{a}] \hat{a}^{\dagger} \rangle \right) \\ &= -\frac{\Gamma}{2} (\bar{n} + 1) \left(+\langle \hat{a}^{\dagger} \hat{a} \rangle + \langle \hat{a}^{\dagger} \hat{a} \rangle \right) - \frac{\Gamma}{2} \bar{n} \left(-\langle \hat{a} \hat{a}^{\dagger} \rangle - \langle \hat{a} \hat{a}^{\dagger} \rangle \right) \\ &= -\Gamma (\bar{n} + 1) \langle \hat{n} \rangle + \Gamma \bar{n} (\langle \bar{n} \rangle + 1) \end{aligned}$$

$$\Rightarrow \boxed{\frac{d}{dt} \langle \hat{n} \rangle = -\Gamma \langle \hat{n} \rangle + \Gamma \bar{n}}$$

$$\text{Solution: } \langle \hat{n} \rangle(t) = \langle \hat{n} \rangle(0) + \bar{n} (1 - e^{-\Gamma t})$$

$$\text{Steady state: } \boxed{\langle \hat{n} \rangle = \bar{n} : \text{Thermal equilibrium}}$$

Coherences:

$$\frac{d}{dt} \langle \hat{a} \rangle = -\frac{i}{\hbar} \underbrace{\langle [\hat{a}, \hat{H}] \rangle}_{\hbar \omega_0 \hat{a}} + \text{Tr}(\mathcal{L}_{\text{relax}}[\hat{\rho}] \hat{a})$$

$$\begin{aligned} \text{Tr}(\mathcal{L}_{\text{relax}}[\hat{\rho}] \hat{a}) &= -\frac{\Gamma}{2} (\bar{n}+1) (\langle \hat{a}^\dagger [\hat{a}, \hat{a}] \rangle + \langle [\hat{a}, \hat{a}^\dagger] \hat{a} \rangle) \\ &\quad -\frac{\Gamma}{2} \bar{n} (\langle \hat{a} [\hat{a}^\dagger, \hat{a}] \rangle + \langle [\hat{a}, \hat{a}] \hat{a}^\dagger \rangle) \\ &= -\frac{\Gamma}{2} (\bar{n}+1) \langle \hat{a} \rangle + \frac{\Gamma}{2} \bar{n} \langle \hat{a} \rangle = -\frac{\Gamma}{2} \langle \hat{a} \rangle \end{aligned}$$

$$\Rightarrow \boxed{\frac{d}{dt} \langle \hat{a} \rangle = \left(-i\omega_0 - \frac{\Gamma}{2}\right) \langle \hat{a} \rangle} \quad \begin{array}{l} \text{Decay} \\ \text{amplitude} \end{array}$$

$$\Rightarrow \boxed{\langle \hat{a} \rangle(t) = \langle \hat{a} \rangle(0) e^{-i\omega_0 t - \frac{\Gamma}{2} t}}$$

Note: The rate of decay is independent of $\langle \hat{a} \rangle$.
How is this possible since the decay of P_{nn} depends on n ?

Look at evolution of coherences of density op

$$\frac{d}{dt} \rho_{n+1, n} = \left(-i\omega_0 - \left[2\bar{n} + \frac{1}{2} + \underbrace{\bar{n}(2\bar{n}+1)}_{\text{dependence on } n}\right] \Gamma\right) \rho_{n+1, n}$$

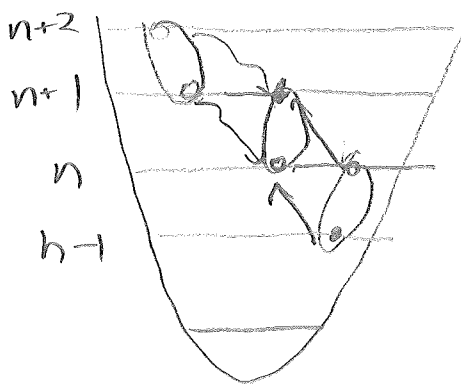
$$+ \sqrt{(n+1)(n+2)} (\bar{n}+1) \Gamma \rho_{n+2, n+1}$$

$$+ \sqrt{n(n+1)} \bar{n} \Gamma \rho_{n, n-1}$$

Note: Coherence decay at rate depending on n

BUT there are also feeding terms

Transfer of coherence



Coherent superposition of $|n+2\rangle$ and $|n+1\rangle$ transferred to superposition of $|n+1\rangle$ and $|n\rangle$

This is only possible because the two decay paths are indistinguishable

This is true only for harmonic ladder

Where the spacing between levels is equal.

Evolution of quadratures

$$\hat{X}_\phi = \frac{\hat{a} e^{-i\phi} + \hat{a}^\dagger e^{i\phi}}{2}$$

$$\Rightarrow \frac{d}{dt} \langle \hat{X}_\phi \rangle = -\Gamma \langle \hat{X}_\phi \rangle \quad (\text{in rotating frame})$$

$$\begin{aligned} \frac{d}{dt} \langle \hat{X}_\phi^2 \rangle &= -\frac{\Gamma}{2} (\bar{n}+1) \left(\langle \hat{a}^\dagger [\hat{a}, \hat{X}_\phi^2] \rangle + \langle [\hat{X}_\phi^2, \hat{a}^\dagger], \hat{a} \rangle \right) \\ &\quad - \frac{\Gamma}{2} \bar{n} \left(\langle \hat{a} [\hat{a}^\dagger, \hat{X}_\phi^2] \rangle + \langle [\hat{X}_\phi^2, \hat{a}] \hat{a}^\dagger \rangle \right) \end{aligned}$$

$$\begin{aligned} \text{Aside: } [\hat{a}, \hat{X}_\phi^2] &= \hat{X}_\phi [\hat{a}, \hat{X}_\phi] + [\hat{a}, \hat{X}_\phi] \hat{X}_\phi \\ &= \hat{X}_\phi e^{i\phi} \end{aligned}$$

$$\begin{aligned} \Rightarrow \frac{d}{dt} \langle \hat{X}_\phi^2 \rangle &= -\frac{\Gamma}{2} (\bar{n}+1) \langle \hat{a}^\dagger e^{i\phi} \hat{X}_\phi + \hat{X}_\phi \hat{a} e^{i\phi} \rangle \\ &\quad - \frac{\Gamma}{2} \bar{n} \langle -\hat{a} e^{-i\phi} \hat{X}_\phi - \hat{X}_\phi \hat{a}^\dagger e^{-i\phi} \rangle \\ &= -\frac{\Gamma}{2} (\bar{n}+1) \left[2 \langle \hat{X}_\phi^2 \rangle + \frac{1}{2} \right] \\ &\quad + \frac{\Gamma}{2} \bar{n} \left[2 \langle \hat{X}_\phi^2 \rangle + \frac{1}{2} \right] \end{aligned}$$

$$= -\Gamma \langle \hat{X}_\phi^2 \rangle + \frac{\Gamma}{4} (2\bar{n}+1)$$

$$\Rightarrow \frac{d}{dt} \langle \Delta \hat{X}_\phi^2 \rangle = \frac{d}{dt} \left(\langle \hat{X}_\phi^2 \rangle - \langle \hat{X}_\phi \rangle^2 \right)$$

$$= -\Gamma \langle \Delta \hat{X}_\phi^2 \rangle + \frac{\Gamma}{4} (2\bar{n}+1)$$

Solution:

$$\langle \Delta \hat{X}_\phi^2 \rangle(t) = \langle \Delta \hat{X}_\phi^2 \rangle(0) e^{-\Gamma t} + (1 - e^{-\Gamma t}) \left(\frac{2\bar{n} + 1}{4} \right)$$

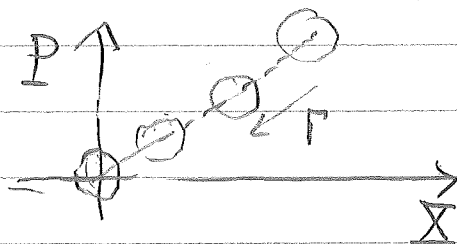
In steady state $\langle \Delta \hat{X}_\phi^2 \rangle = \frac{1}{4} (2\bar{n} + 1)$

Note: Even at zero temperature, the vacuum with $\bar{n} = 0$, the quadrature fluctuations damp

$$\langle \Delta \hat{X}_\phi^2 \rangle(t) = \langle \Delta \hat{X}_\phi^2 \rangle(0) e^{-\Gamma t} + \frac{1}{4} (1 - e^{-\Gamma t/2})$$

• Example: Coherent State $\langle \Delta \hat{X}_\phi^2 \rangle(0) = \frac{1}{4}$

$$\Rightarrow \langle \Delta \hat{X}_\phi^2 \rangle(t) = \frac{1}{4} \quad \forall t$$

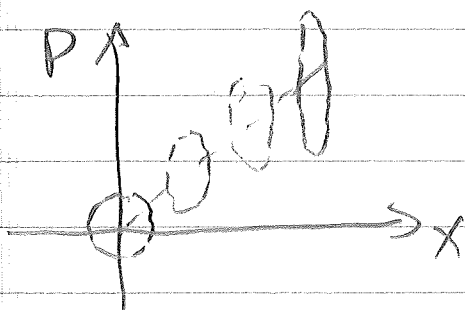


For a coherent state mean amplitude damps but fluctuations unchanged

Coherent state is eigenstate of Lindblad operator

$$\hat{L} = \sqrt{\Gamma} \hat{a} \Rightarrow \text{"pointer state"}$$

• Example Squeezed State $\langle \Delta \hat{X}^2(0) \rangle = \frac{e^{-2r}}{4}$



$$\langle \Delta \hat{P}^2(0) \rangle = \frac{e^{+2r}}{4}$$

Damping kill SQUEEZING

Squeezed state depend on correlated photons.

Fokker - Planck Eq. for quasi-probability funct

An important tool for dealing with the damped SHO is to consider the quasi-probability distributions, for example the Wigner funct.

Using characteristic fn $\chi(\beta) = \text{Tr}(\hat{\rho} \hat{D}(\beta))$

$$\Rightarrow \frac{\partial \chi(\beta)}{\partial t} = \text{Tr} \left(\frac{d\hat{\rho}}{dt} \hat{D}(\beta) \right)$$

$$\Rightarrow \frac{\partial W(\alpha)}{\partial t} = \frac{1}{\pi^2} \int d\beta \frac{\partial \chi(\beta)}{\partial t} e^{\alpha\beta^* - \alpha^*\beta}$$

In terms of the quadratures:

$$\frac{\partial}{\partial t} W(x, p, t) = -\frac{\Gamma}{2} \left(\frac{\partial}{\partial x} x + \frac{\partial}{\partial p} p \right) W(x, p, t) + \frac{\Gamma}{4} (2\bar{n} + 1) \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial p^2} \right) W(x, p, t)$$

This is known as the Fokker-Planck equation.

- The first term leads to "drift" on the mean phase-space position to the origin, at rate Γ
- The second term leads to "diffusion" of the distribution, spreading in steady state

The F-P equation preserves Gaussians
If W is Gaussian @ $t=0$

$$\text{then } W(x, p, t) = \frac{1}{2\pi \Delta X(t) \Delta P(t)} e^{-\frac{x^2}{2\Delta X^2(t)}} e^{-\frac{p^2}{2\Delta P^2(t)}}$$

Damping vs. Decoherence

So far our examples have shown that the master equation for the damped SHO have lead to damping and diffusion. These ~~is~~ are a "classical" phenomena; familiar in stochastic processes (e.g. Brownian motion). The coupling of a quantum system to an environment (reservoir) also leads to very nonclassical effects.

Decoherence of a "Schrodinger Cat"

Suppose we start at $t=0$ with a "macroscopic superposition" of two coherent states

$$|\Psi(0)\rangle = \frac{1}{\sqrt{2}} (|\alpha_0\rangle + |-\alpha_0\rangle) \quad \text{E.g. } \alpha_0 \text{ real}$$

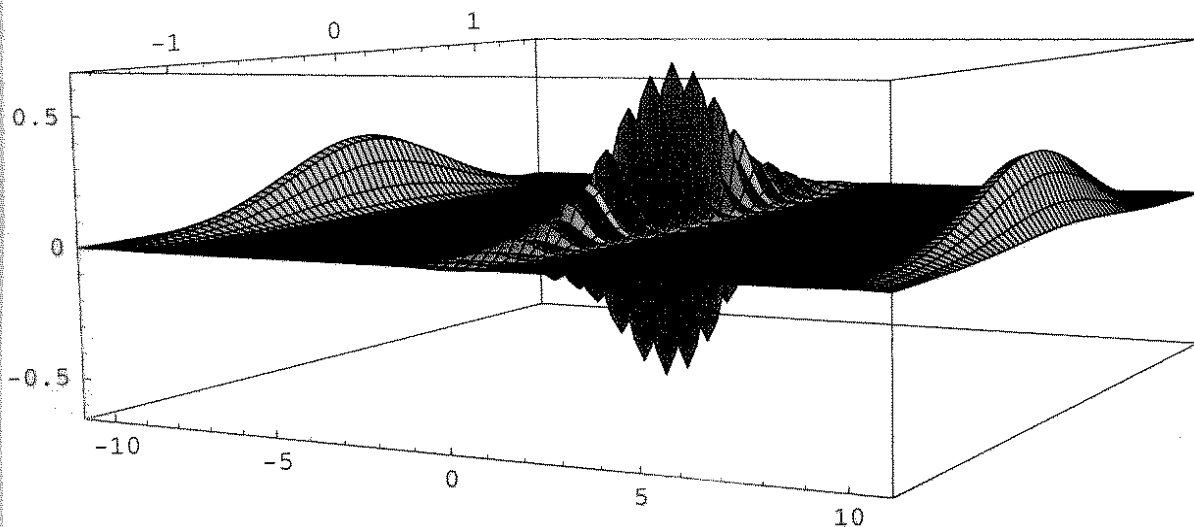
$$\text{where } \alpha_0 = X_0 = \left(\frac{X_0}{\sqrt{\frac{\hbar}{2m\omega_0}}} \right) \quad \text{For mechanical oscillator}$$

$$\text{Macroscopic} \Rightarrow X_0 \gg 1$$

We found the Wigner function in the Problem Set

$$W(x, p, 0) = \frac{C}{2} \left[W_+(x, p) + W_-(x, p) + \frac{4}{\pi} \cos(4X_0 p) e^{-2(x^2 + p^2)} \right]$$

$$\text{Where } W_{\pm}(x, p) = \frac{2}{\pi} e^{-2\left\{ (x - X_0)^2 + p^2 \right\}}$$



The Wigner function shows localized Gaussians at $x = \pm X_0$, $p = 0$. In addition it shows rapid oscillations in p at a rate depending on X_0 . These come from the "coherence", that is a statistical mixture of coherent states

$$\hat{\rho} = C \left(\frac{1}{2} |\alpha_0\rangle \langle \alpha_0| + \frac{1}{2} |-\alpha_0\rangle \langle -\alpha_0| \right)$$

$$\Rightarrow W(x, p) = C \left(\frac{W_+(x, p)}{2} + \frac{W_-(x, p)}{2} \right)$$

The "interference in phase space" distinguishes the statistical mixture from the "Schrödinger cat" which is in a superposition of two macroscopic possibilities.

However, diffusion in the FP equation very quickly wipes out the ~~the~~ coherence for macroscopic superpositions

The oscillating term $\cos(2pX_0)$ under diffusion in p :

$$\Gamma \left(\bar{n} + \frac{1}{2}\right) \frac{\partial^2}{\partial p^2} \cos(2pX_0) = \Gamma \left(\bar{n} + \frac{1}{2}\right) X_0^2 \cos(2pX_0)$$

\Rightarrow Amplitude of oscillation decay at rate

$$\gamma_{\text{coh}} = \Delta_0^2 \bar{n} \Gamma = \Gamma \left(\frac{X_0^2}{\hbar / 2m\omega} \right) \bar{n}$$

In high temperature limit $\bar{n} = \frac{kT}{\hbar\omega}$

$$\Rightarrow \gamma_{\text{coh}} = \Gamma \left(\frac{X_0^2}{\frac{\hbar^2}{2mkT}} \right) = \Gamma \left(\frac{X_0}{\lambda_{\text{DB}}} \right)^2$$

where $\lambda_{\text{DB}} = \frac{\hbar}{\sqrt{2mkT}}$ is the thermal deBroglie wavelength

The lifetime of coherence

$$\tau_{\text{coh}} = \frac{1}{\gamma_{\text{coh}}} = \tau_{\text{decay}} \left(\frac{\lambda_{\text{DB}}}{X_0} \right)^2 \quad \text{where } \tau_{\text{class}} = \frac{1}{\Gamma}$$

For example, at room temperature, $m = 1 \text{ gram}$
 $X_0 = 1 \text{ cm}$

$$\Rightarrow \frac{\tau_{\text{coh}}}{\tau_{\text{decay}}} = 10^{-40}$$

Even if $\tau_{\text{decay}} = 10^{17} \text{ s}$
 (age of universe)
 $\tau_{\text{coh}} \sim 10^{-23} \text{ s}$